Abstract.
Spiral Fresnel wavelet patterns carrying topological charges allow for a holographic en/decoding of spin and charge. The nature of charge is given by a linear spin-precession coupling encoded in the spiral form with (Dirac,...) magnetic monopoles located at the spiral phase singularities. The Berry phase pattern emerging in the overlap region between charged spiral zones on the Poincaré disc are perfect Fresnel holograms, wavelets, or “Fresnelets” representing neutral particles (standard Fresnel zones). The corresponding stereographic projections of rotated rotations in $N$-dimensional space onto the $N$-1-dimensional Poincaré disc show an exact algebraic relation to the Kepler/Coulomb-Oscillator duality. The spherical results can be related by analytic continuation to the hyperbolic case of helicoid/catenoid modes (DNA,...) supported by solutions to the Schrödinger equations with a superintegrable Hamiltonian and hypergeometric functions from Pöschl-Teller potentials in the angular variables. The corresponding bridge between spaces of different curvature in angular variables is given by the Mercator-Gudermann stereographic relations mapping loxodromes to helicoids. Coupling strengths fit very well to the Sommerfeld fine structure constant, where the deviation from 1/137 can be assigned to the Berry phase. Regarding multi-monopole interaction, the relevance to strong interaction is show with a model called $^9$Be-Berryllium, where 4 charged (spiral) patterns excite and support 5 neutral patterns in the overlap region at a similar scale. Above is a computer-generated gradient picture of the $^{12}$C-Carbon model pattern.
1. Stereographic Projection of Rotated Rotations and Differential Relations

The Rolling Rotation of the Rotator
Given three Euler angles \( \varphi, \theta, \kappa \), a special holonomic condition (characterizing a rotated rotation like a rolling cone) with interesting properties providing for a Berry geometric phase on a spherical surface is given by \([4, 5, 6]\) (curvature index \( \kappa \))

\[
\gamma_{\text{Berry}} = \varphi - \theta = \varphi \left( 1 - \cos \theta \right) \iff \frac{\theta}{\varphi} = \cos \theta. \quad (1)
\]

Mapping this (rolling) rotation \( \varphi \) of the rotator axis onto a cylinder by a Mercator stereographic projection corresponds to a Gudermannian relation

\[
\cos \theta_x = \frac{1}{\cosh \varphi_x} = \frac{d\theta_x}{d\varphi_x}, \quad (2)
\]

where the axis tilt angle \( \theta_x \) is mapped onto the height of the cylinder \( \varphi_x \) \([7]\). With \( \varphi \) linearly related to the hyperbolic height \( \varphi_x \) by a helical pitch \( M \), we get

\[
\varphi = \varphi_x M^{+\kappa}, \quad (3)
\]
a helix projected to a loxodrome on the sphere, see fig 1. Eqs. (1)–(3) lead to

The Kepler/Coulomb - Oscillator relation (KC-O) \([2]\)

\[
e^{-M^{+\kappa} \theta_x/\theta} \propto \cosh \varphi_x = \varphi_x M^{+\kappa} = \frac{d\varphi_x}{\theta_x}, \quad (4)
\]

well known since Schrödinger. Here it describes a gyroscopic dissipation of holonomy, where a negative exponential loxodromic oscillation of the spin axis on the sphere is mapped to a helical path on the cylinder with Kepler-Coulomb \( 1/\varphi_x \) radial dependence. The optimum distance and magic precession \( \theta_x \) can be iteratively obtained for a given \( M \).

Pöschl-Teller (PT) and Sine-Gordon Nonlinear Potentials
The KC-O is based on the pull-pack (4) applied to the PT Schrödinger equation \([2]\) with nonlinear potentials \( V_\pm(\theta_\pm) \) and hypergeometric \( \Psi_\pm \)

\[
\frac{d^2 \Psi_\pm}{d \theta_\pm^2} + V_\pm \Psi_\pm = E_\pm \Psi_\pm, \quad V_\pm = \frac{a}{\sin^2 \left( \sqrt{\mp \kappa} \theta_\pm \right)} + \frac{b}{\cos^2 \left( \sqrt{\mp \kappa} \theta_\pm \right)}, \quad (5)
\]
narrowing the magic angular range, handling both curvatures by analytic continuation. This form can directly be obtained considering the SO(3) and SO(2,1) rotations subject to equ. (1), where \( \Psi_\pm \) represent the Wigner rotation-transition matrix elements \([7]\). The hyperbolic SO(2,1) solution has been assigned to the DNA helix structure \([1]\) as a minimal surface in accordance with equ. (3), which means that a stereographic projection (2) is part of the Hamiltonian. The Gudermannian (2) can be written as

\[
\frac{d^2 \theta_x}{d \varphi_x^2} = - \frac{1}{2} \sin \left( \sqrt{\mp \kappa} 2 \theta_\pm \right) \quad (6)
\]

showing a sine/sinh Gordon-Bäcklund type solitonic local twist.
Increasing Dimensionality, Curvature-Invariance

We can add to the 1-dimensional Gudermann relation another complementary Gudermannian. This provides a 2x2-dimensional (vector) cross-relation on geodesic surfaces with different curvature signs, leading to the curvature-invariant \((+K \leftrightarrow -K)\) forms

\[
\cos \sqrt{-K} \theta_{-K} = \frac{1}{\cos \sqrt{K} \varphi_{+K}} \quad \text{and} \quad M^{+K} \theta_{+K} = \cos \sqrt{+K} \theta_{+K},
\]

\[
\tan(\pm \sqrt{K} \theta_{\pm K}) = \sin(\pm \sqrt{K} \varphi_{\mp K}), \quad \sin(\pm \sqrt{K} \theta_{\pm K}) = \tan(\pm \sqrt{K} \varphi_{\mp K}),
\] (8)

Magic Angle Precession if \(\varphi_{\pm K} = j \pi\) [4-7]. The closed/recurrent geodesic flow between metric spaces of different constant curvature will be relevant to holographic encoding, where it is necessary to have also a proper decoding process to get the information back. Note, we get for \(M = 137\) and one loop \(\varphi_{+K} = \pi\) on the sphere an accurate value for the Sommerfeld fine-structure constant \(\alpha = \theta_{+K} \approx 137.036\) , where the Berry phase can be assigned to the deviation from the integral monopole quantum number \(M\). Considering the angular length ratio we get a special kind of isometry confirming eqs. (1) and (3), see figs. 3, 4: if one angular dynamic drives the other (precession as a gauge potential driving spin currents) the invariant angular ratio provides for a good transmission of spin current between spaces of different curvature, see fig. 4, maybe relevant to twister dynamics.

**N-dimensional Stereographic Projection**

In ambient Weierstrass coordinates \((x_0, x) = (x_0, x_1, ..., x_N) \subset \mathbb{R}^{N+1}\) [3] representing spherical or hyperbolical surfaces \(x_0^2 \pm K x^2 = 1\) we can encode the (holographic) depth coordinate \(x_0\) by a stereographic projection to Poincaré space coordinates \(q \in \mathbb{R}^{N}\), with

\[
x = \frac{2q}{1 \pm Kq^2}, \quad x_0 = \frac{1 \mp Kq^2}{1 \pm Kq^2}, \quad ds^2 = \frac{4dq^2}{(1 \pm Kq^2)^2}, \quad (9)
\]

an inversion in \(x_0\), and radial distance in both systems given by

\[
R_{\pm K} = \sqrt{x^2} = \frac{2\sqrt{q^2}}{1 \mp Kq^2}, \quad \sqrt{q^2} = \tan\left(\frac{\sqrt{\pm K} \theta_{\pm K}}{2}\right), \quad \sqrt{x^2} = \tan\left(\frac{1}{2} \sqrt{\pm K} \theta_{\pm K}\right).
\]

\(R_{\pm K}\) is the common variable in parts of the superintegrable \(N\)-dim. Hamiltonian [3] and maps between KC, oscillator \(R_{\pm K}^2\), monopole, and PT-potential terms from inversion by switching curvature

\[
H = T + \frac{\mu^2}{2R_{\pm K}^2} + \frac{p^2}{2R_{\pm K}^2} \sum_{i=1}^{N} \frac{b_i}{q_i^2} + \frac{\alpha}{R_{\pm K}}, \quad R_{\pm K}^{+K} = R_{-K}^{-K}. \quad (10)
\]

The mass scalar part of the kinetic energy term \(T\) is given by

\[
m_{\pm K} = \lambda_{\pm K}^2, \quad \lambda_{\pm K}^2 \varphi^2 R_{\pm K}^2 = \gamma_{\text{Berry}}^2 q^2, \quad \varphi = \frac{p}{m_{\pm K}}, \quad \alpha = \left(1 - x_0^2\right) = \frac{\gamma_{\text{Berry}}^2}{\varphi^2}. \quad (11)
\]
2. Charged Geometric Phase Patterns and Holography

Charged Spiral Zones (FSZ) carry Berry Phase Patterns

A diffraction pattern carrying holographic information encoded by stereographic projection according to eqs. (9)-(10) can be thought as a coherent superposition of many point holograms or point diffractions, which are the Fresnel zone patterns on the curved \( x \) or flat \( q \)-surface and image (hyper)plane. The diffraction patterns arise by interference between flat and spherically propagating wave signals, usually due to Huygens’ principle. Spin and topological charges can be easily introduced by considering twisted patterns also called spiral zone plates, see figs. 5 - 7. The holographic depth \( x_0 \) is the cosine of \( \theta_{\kappa} \) the scattering angle and provides with eqs.(1)-(3) and (9) for an interference phase pattern \( \Psi'(x_0, x) \) on the flat \( q \)-hypersurface at \( 1 + x_0 \) given by

\[
\Psi'(x_0, x) \propto e^{i\varphi}, \quad \gamma_{\text{Berry}} = \varphi \frac{k x^2}{1 + x_0} = M \left[ \varphi - \gamma_{\text{Berry}} = M \arctan \left( \frac{x_1}{x_2} \right) \right],
\]

(15)
carrying \( M \) monopole charges

\[
\theta_{\kappa} = \varphi - \gamma_{\text{Berry}} = M \arctan \left( \frac{x_1}{x_2} \right).
\]

(16)

Fig 6: Left the ideal circular symmetric Fresnel zones with topological charge \( M = 0 \), in the mid and right charged spiral zones FSZ with \( M = 1 \) and 2 (double helix), respectively.

see figs. 5-7, where the twist \( \varphi \) is the sum of a geometric \( \gamma_{\text{Berry}} \) and a dynamical twist \( \theta_{\kappa} \). Eq. (15) has the well known form with \( M \) as the number of topological monopole charges and (15) is exact - no approximation. Reversely, the point hologram acting as a Fresnel lense can focus a plane wave back onto a point located at a depth \( x_0 \), generating extreme local field densities, spin with topological charges. For \( q = \{x_1, x_2\}, x = \{x_0, x_1, x_2\} \) there are highly accurate computer-generated diffraction patterns by convolution techniques in Fourier space based on the Berry phase propagator in eq.(15), see figs. 5-10.

Fig 5: Stereographic projection mapping spin angular variables to \( \Psi'(x_0, x) \). Fresnel (spiral) zone holography requires a coherent reference beam generating (twisted) paraboloid \( x^2 \) interference patterns [7].

Fig 7: FSZ with \( M = 18 \) showing a central hole.
Neutral Berry Patterns from Overlapping Charged Patterns

A linear superposition of the patterns provides for information about interaction. A simple computer experiment shows that the interaction itself is neutral with \( M = 0 \), where two nearby located parabolic FSZ patterns carrying the same charge/twist are overlapped, see fig. 8. Shifting two parabolic diffraction centers apart at distance \( 2x_1 \) can be be assigned to a virtual neutral pattern with similar size/energy

\[
x^2_{\text{separation}} \rightarrow \frac{1}{2}(x - x_1)^2 + \frac{1}{2}(x + x_1)^2 = x^2 + x_1^2.
\]

(17)

Berryllium \(^9\)Be

Assume that neutral (Berry) soliton modes emerge and mediate in the overlap region between charged or twisted geometric phase patterns, we could assign this behaviour to the role of neutrons/neutrinos in strong and weak interactions, e.g. Berryllium has only one stable configuration with 5 neutrons \(^{9}\)Be and a very rare isotope with 3 neutrons, see fig. 10.

More

The presented projections can be directly related to known Hopf fibrations and various types of monopoles in higher dimensions on a rather abstract level, where recurrent phase relations are missing. Interestingly, Berry phase patterns also apply to molecular (DNA) structures sharing the same bifurcation instabilities or overload ambiguities, with first bifurcation at \([1,5]\) \( \theta_{1x} \) \( \tanh \sqrt{\pm 1} \theta_{1x} = \sqrt{\pm 1} \).

Figs 10: Berryllium

There are 2 possibilities partly overlapping 4 identical charged patterns. Top: 5 neutral patterns partly overlapping with a high mean square reduction. Right: the chain configuration has 3 neutral patterns but lower stability (the gradient is shown). Here Helium is a kind of sub-structure.

References (see also www.quanics.com)