Geodesic Holonomy Attractor between Surfaces of Different Curvature Signs relevant to Spin Transport

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Abstract. We will consider nonlinear holonomy effects -especially the spin dissipation dynamics- arising in the transport of a linear rotator between metric spaces with different curvature (positive, zero, negative). The extra 3D spin vector current induced by curvature or metric distortion provides for a holonomic attractor called "Magic Angle Precession" (MAP). Limitations and instabilities of the spin current exchange are assigned to bifurcations at high precession loads as the driving gauge potential. In the classical range the chaotic dynamics can be verified with a mechanical toy gyroscope with built-in spin-precession coupling that could also be modeled by a Chua-type electronic circuit. Transporting vector currents composed by spin and precession is treated by Schwarz-Christoffel triangle conformal maps with constant Schwarzian derivative and hypergeometric monodromy. Handling both curvatures simultaneously as a metric distortion is possible by a composite hypergeometric mapping function linearly composed as a ladder/recurrence relation between dual hypergeometric functions related by inversion. This leads to the well known Schrödinger hypergeometric quantum mechanical solution providing for Pöschl-Teller type potentials, quantization, factorization, and ladder operators. By pull-back we get the generalized Gauss linking number density differential form.
Keywords: holonomy, chaotic precession, geometric phase, hypergeometric, curvature, Berry, Chua, quantum gravity, Schrödinger, Pöschl - Teller, Legendre, Gegenbauer, Gauss, linking number, spin, magic angle spinning.

1 Introduction

We will analyze the chaotic precession dynamics of rotators transported in and between hyberbolic, flat, and spherical spaces, where we get an extra 3D vector rotation induced by holonomy, known as geodetic precession from parallel transport on curved paths. Adding a few extra coupling or damping terms to rotated spin systems, which is typical for a real system, can lead to dissipation induced instabilities or bifurcations, where it is rather expensive to actively control chaotic motion to obtain the sufficient stability conditions at the equilibrium points even for relatively simple gyroscope systems [6]. Transporting rotatable rotators (gyroscopes [9]) like spinning electrons over metric distortions in curved space or on curved Bloch surfaces can result in a highly complex chaotic behavior due to nonlinear holonomy effects in periodic loops [15]. In the classical limit we have no strong restrictions regarding the number and availability of states, but in the periodic or quantum case (standing holonomy waves) there are phase boundary conditions. In this paper we will provide for a general quantum transmission matching condition, previously called 'Magic Angle Precession" (MAP) supporting the transport of a linear rotator between metric spaces with different curvature [2]. We are looking for spin/precession preconditions given by or adjusted to a curvature step or metric distortions allowing for a lossless spin current with minimum reflection, where precession as a gauge field is driving the spin current.

2 Rotator Mediating or Parallel Transported between Surfaces with Different Curvature

Taking θ as the precession angle we assume that the rotator is parallel transported between surfaces with different curvature, where precession from holonomy can be classified according to the curvature sign (-1,0,+1). Gyroscopic precession corresponds to the spherical case (subscript $_s$, curvature +1) [9], Thomas precession (well known from special relativity) to the hyperbolic case (subscript $_h$, curvature -1) [10]. Both situation can be simultaneously treated by taking conformal mappings onto the flat Poincaré disc (curvature 0), where we consider the real arc lengths given by the integrals $\theta_s = \int |d\theta_s|$ and $\theta_h = \int |d\theta_h|$ on the compact Riemann surfaces with conformal metrics

$$|d\theta_s| = \frac{2\,|d\xi_s|}{1+|\xi_s|^2}, \quad |d\theta_h| = \frac{2\,|d\xi_h|}{1-|\xi_h|^2}\ , \tag{1}$$

ξ_s or ξ_h are the complex variables on the Poincaré disc, $r_s = |\xi_s|$ and $r_h = |\xi_h|$ are the radial distances, in spin dynamics often referred to as rapidity. From Poincaré we know that we have the full group of isometries on this disc and also the sense preserving Möbius transformations $PSL(2,R)$ [14]. On the Poincaré disc with a given center and scale we will first consider the arc length ratio between hyperbolic, flat, and spherical cases. With rotations measurable as precession exclusively induced by holonomy (which can be regarded as metric distortions [5], see below) we focus on the Möbius inversion between spherical and hyperbolic space as the fractional linear transformation connecting the conformal metrics in eq.(1). The correspondent distance inversion invariant can be written as a the differential product

$$d\theta_s d\theta_h = dr_s dr_h = 1, \tag{2}$$

where the arc length differential relation can be extended to an arc length integral length relation. Eqs.(2) and (1) are fulfilled by the rapidity relation [2]

$$\frac{1}{1-r_h^2} = \frac{r_s^2}{r_h^2} = 1 + r_s^2, \tag{3}$$

that provides for

$$r_h = \sin\theta_s = \tanh\theta_h, \tag{4}$$

and

$$r_s = \sinh\theta_h = \tan\theta_s. \tag{5}$$

With invariant arc lengths and length inversion $r_h r_s = \theta_h \theta_s$ providing for

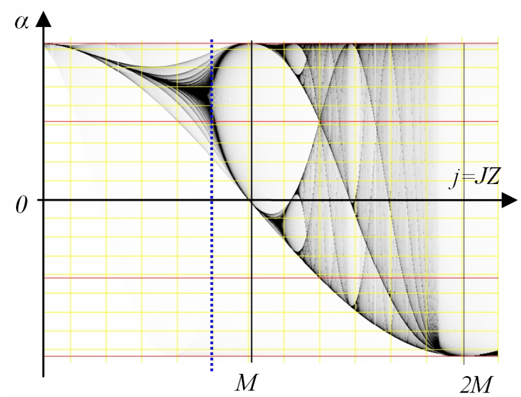

Fig. 1. MAP phase space density bifurcation diagram with $r_s = \pi j$ and $\theta = j\alpha$. Smearing effects are introduced due to phase fluctuations.

a mapping condition, the measure characterizing the distortion related to precession will be given by the ratio

$$f(\xi) = \frac{\int |d\theta_h|}{\int |d\xi_s|} = \frac{\theta_h}{r_s} = \frac{\int |d\xi_h|}{\int |d\theta_s|} = \frac{r_h}{\theta_s}. \tag{6}$$

Eqs.(4) - (6) provide for the θ_s and θ_h MAP conditions [4], [3]

$$\frac{r_h}{r_s} = \cos\theta_s = \frac{f\theta_s}{r_s}, \tag{7}$$

$$= \frac{1}{\cosh\theta_h} = \frac{f r_h}{\theta_h}. \tag{8}$$

f will represent coupling induced by a metric distortion leading to precession that can be expressed by a winding number M. For a given constant winding $f r_h \propto M$ in eqs.(7) and (8) (probably due to a periodic boundary or quantum condition) we are confronted with a chaotic attractor given by the cosine and hyperbolic cosine map [3] with phase space density bifurcation diagram shown in Figure 1.

3 Spin Triangulation and Mapping of Composite Spaces by Hypergeometric Functions

If we want to connect the hyperbolic and spherical precession metrics by a mapping function in accordance with eqs.(7) and (8), spin and precession will have a triangular vector relation providing for the invariant total angular momentum on the tangential path of parallel transport. The curvature change be approached as a metric distortion f acting on triangular vector relation [2], [5]. We could choose the situation, where two out of three angles are identical ($A_s = A = A_h$, $B_s = B = B_h$), whereas the third angle $C_h < C < C_s$ is different due to holonomy or metric distortion, see Figure 2. Using well-

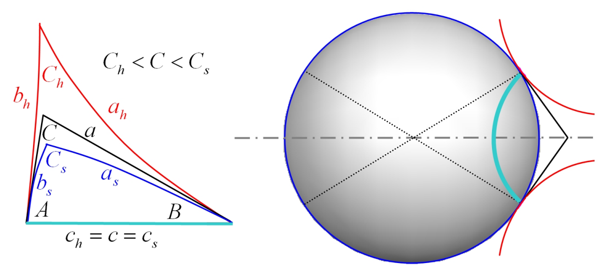

Fig. 2. The arc lengths of triangles of locally parallel curved surfaces (right) projected onto the Poincaré disk (left). Hyperbolic is red, flat is black, spherical is blue with curvature $-1, 0, +1$, respectively. The triangles are locally aligned on the common arc $c_h = c = c_s$ (green), where two out of three angles are identical $A_s = A = A_h$, $B_s = B = B_h$.

known expressions for the triangle functions we arrive with invariant ratio eq.(6) at eqs.(7) and (8)[2]. In [5] the metric distortion associated to a triangle map f is the ratio of the euclidean length element $|d\theta|$ at $\theta = f(z)$ to the spherical length element $2|dz|/(1+|z|^2)$ at z, or to the hyperbolic length element $2|dz|/(1-|z|^2)$ at z, depending on whether we have a spherical or hyperbolic triangle. This means, that the precession of the rotator in eq.(1) just measures the metric distortion. Considering both cases separately it is easy to find conformal mappings leaving the angles invariant. But we are more interested in the composite case, where metrics of different curvatures are tangentially aligned and the change in the sum of angles is changed by the metric distortion (holonomy effect leading to precession) [2]. Considering the pure case, the Riemann mapping theorem provides for a conformal mapping sending the triangle vertices to the points $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$, which are the singularities of a second order ordinary differential equation (ODE2). The algebraic solutions are given by the Gauss hypergeometric function $f(z)$

[13] obeying the hypergeometric differential (Fuchsian) equation [1]

$$z(1-z)\frac{d^2f(z)}{dz^2}+\left[c-(a+b+1)z\right]\frac{df(z)}{dz}-abf(z)=0. \tag{9}$$

The hypergeometric Gauss equation (9) providing for $f(z) = {}_2\mathrm{F}_1\left(\begin{smallmatrix} a,\, b \\ c \end{smallmatrix}\middle|\, z\right)$ has three parameters a, b, c related to the (precession) singularities $\theta = \{\theta_1, \theta_2, \theta_3\}$ at $z = \{p_1, p_2, p_3\}$ by

$$\begin{aligned} \theta_1 &= 1-c, \\ \theta_2 &= c-a-b, \\ \theta_3 &= a-b\ . \end{aligned} \tag{10}$$

The holonomy change due to a metric distortion can be measured as a change in the sum of angles in a triangle from eq.(10) $H = 1 - \sum \theta_i = 2b$. This fact appears to be incompatible with the concept of conformal mappings leaving the angles invariant. Handling both curvatures simultaneously as a metric distortion is possible by a composite Gauss hypergeometric function $f_k(z)$ with Gauss recurrence relation as a sum of two weighted parts $f_i(z)$ and $f_j(z)$ written as $H_k f_k(z) = H_i f_i(z) + H_j f_j(z)$, where the individual holonomies are related by $H_k = H_i + H_j$. We are especially interested in the composite situation based on two components $f_i(z)$ and $f_j(z)$ related by inversion according to eq.(6). The recurrent situation leads to quantum ladder operations found by Schrödinger, see below. Consider the hypergeometric holonomic decomposition of the composite holonomy $H_k = 2b = 2\lambda - 2\epsilon - 2$ by a hypergeometric modulation into a negative $H_j = -2\epsilon - 1$ and a positive part $H_i = 2\lambda - 1$. The correspondent hypergeometric decomposition is given by [1]

$$\begin{aligned} H_k f_k(z) &= H_i f_i(z) + H_j f_j(z) \\ &= (2\lambda-2\epsilon-2)\, f_k(z) = (2\lambda-1)\, f_i(z) + (-2\epsilon-1)\, f_j(z) \\ = (2\lambda-1)\, {}_2\mathrm{F}_1\left(\begin{matrix} \lambda+\epsilon,\ \lambda-\epsilon-1 \\ 2\lambda-1 \end{matrix}\,\middle|\, z\right) &- (2\epsilon+1)\, {}_2\mathrm{F}_1\left(\begin{matrix} \lambda+\epsilon,\ \lambda-\epsilon-1 \\ \lambda+\frac{1}{2} \end{matrix}\,\middle|\, z\right) \\ &= (2\lambda-2\epsilon-2)\, {}_2\mathrm{F}_1\left(\begin{matrix} \lambda+\epsilon,\ \lambda-\epsilon \\ \lambda+\frac{1}{2} \end{matrix}\,\middle|\, z\right)\ . \end{aligned} \tag{11}$$

Inversion as a conformal mapping corresponds to the exchange of the triangle vertices at 0 and ∞, a switch of the hypergeometric singularities leaving the singularity at 1 invariant. Computing the vertices angles of the hypergeometric functions in the decomposition of eq.(11) according to eq.(10) we get

$$\begin{aligned} \theta_{i1} &= \tfrac{3}{2}-\lambda, & \theta_{j1} &= \tfrac{1}{2}-\lambda, & \theta_{k1} &= \tfrac{1}{2}-\lambda, \\ \theta_{i2} &= \tfrac{1}{2}-\lambda, & \theta_{j2} &= \tfrac{3}{2}-\lambda, & \theta_{k2} &= \tfrac{1}{2}-\lambda, \\ \theta_{i3} &= 2\epsilon+1\ , & \theta_{j3} &= 2\epsilon+1\ , & \theta_{k3} &= 2\epsilon\ . \end{aligned} \tag{12}$$

We can see in eq.(12) the switch in $\theta_{i1} = \theta_{j2}$ and $\theta_{j1} = \theta_{i2}$, revealing the inversion relation with invariant $\theta_{i3} = \theta_{j3}$. Obviously, the singularities of $f_k(z)$ and $f_i(z)$ provide for the same angular sum. First we will show that there are additional constraints given by the Schwarzian criterion of conformal mapping. Later we will show that the holonomy weights or metric distortion amplitudes $H_k = H_i + H_j$ are just quantum numbers from quantization and factorization.

4 The Schwarzian Derivative

The general context of Hypergeometric functions can be applied to solve ordinary differential equations by symbolic computing of the Fuchsian form [1]

$$f'' + p(\alpha)f' + qf = 0, \quad \frac{df}{d\alpha} = f',\ \frac{d^2}{d\alpha^2} = f'',\ q = -ab = \pm\omega_q^2,\ q' = 0, \quad (13)$$

with $p, q \in \mathbf{C}(\alpha)$. As a simple test, we get the MAP functions eqs.(7) and (8) with $\alpha_s = \theta_s/2$ and $\alpha_h = \theta_h/2$

$$f_h \propto \frac{M\alpha_h}{\cosh\alpha_h},\ p_h = \tanh\alpha_h, \quad f_s \propto \frac{\cos\alpha_s}{M\alpha_s},\ p_s = \frac{2}{\alpha_s}. \quad (14)$$

. Possible $p, q \in \mathbf{C}(\alpha)$ can be obtained directly from the Schwarz derivative [14]. Let $s \in \mathbf{C}(\alpha)$ denote a quotient of two distinct solutions of the hypergeometric (Gauss) equation (9) and call it a *Schwarz map* for the hypergeometric equation, then one important criterion is given by the *Schwarzian derivative* [5]

$$\{s, \alpha\} = \left(\frac{s''}{s'}\right)' - \frac{1}{2}\left(\frac{s''}{s'}\right)^2 = \frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 = 2q - p' - \frac{1}{2}p^2, \quad (15)$$

which is invariant under all linear fractional transformations, where all closed paths with a common fixed point give a group of fractional-linear transformations that acts on the branches of s generating the metric (1). The Schwarzian derivative is a criterion for the existence and monodromy of a compact Riemannian metric of constant curvature and a criterion for a proper pull-back transformation. With constant singularities $\theta_j' = 0$ we have with $q' = 0$ from eq.(15)

$$\{s, \alpha\}' = -p'' - \frac{1}{2}(p^2)' = 0,\ p = -\frac{p''}{p'}. \quad (16)$$

leading to

$$p \propto \tan(\omega_q\alpha),\ p \propto \cot(\omega_q\alpha)\ ,\ p \propto \tanh(\omega_q\alpha),\ p \propto \coth(\omega_q\alpha)\ ,$$
$$p \propto 2/\alpha\ . \quad (17)$$

$\xi = f(z)$	$p(\alpha)$	pullback z	a	b	c	context
$(\pi-\alpha)/\cos(\omega_q\alpha)$	$-2\lambda\tan(\omega_q\alpha)$	$(1+t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}+\lambda$	MAP
$(\pi-\alpha)/\sin(\omega_q\alpha)$	$2\lambda\cot(\omega_q\alpha)$	$(1-t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}+\lambda$	Linking Number
$(\pi-\alpha)/\cos(\omega_q\alpha)$	$-2\lambda\tan(\omega_q\alpha)$	$(1-t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}$	MAP
$(\pi-\alpha)/\sin(\omega_q\alpha)$	$2\lambda\cot(\omega_q\alpha)$	$(1+t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}$	Linking Number
$(\pi-\alpha)/\cosh(\omega_q\alpha)$	$2\lambda\tanh(\omega_q\alpha)$	$(1+t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}+\lambda$	MAP
$(\pi-\alpha)/\sinh(\omega_q\alpha)$	$2\lambda\coth(\omega_q\alpha)$	$(1-t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}+\lambda$	Linking Number
$(\pi-\alpha)/\cosh(\omega_q\alpha)$	$2\lambda\tanh(\omega_q\alpha)$	$(1-t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}$	MAP
$(\pi-\alpha)/\sinh(\omega_q\alpha)$	$2\lambda\coth(\omega_q\alpha)$	$(1+t)/2$	$\lambda\pm\epsilon$	$\lambda\mp\epsilon$	$\frac{1}{2}$	Linking Number

Table 1. A table listing the hypergeometric solutions to $\xi'' + p(\alpha)\xi' + q\xi = 0$ with $\theta = 2\omega_q\alpha$, $q = -\omega_q^2 = \epsilon^2 - \lambda^2$, $t = \cos\theta$.

From the metrics (1) with $r = |\xi|$ we get the real transcendental pull-back relations

$$z_s = \frac{1}{1+r_s^2} = \frac{1+t_s}{2} = \frac{1}{z_h}\ , \quad z_h = \frac{1}{1-r_h^2} = \frac{1+t_h}{2}\ , \tag{18}$$

providing for the transcendental mappings

$$t_s = \cos\theta_s,\ t_h = \cosh\theta_h. \tag{19}$$

From now on we ommit subscript $_h$ and $_s$, which can be directly found from the hyperbolic or spherical type of the ODE2, see [1] and the list in Table 1.

5 Schrödinger Quantization and Factorization

Since $f_k(z)$ is a solution to a ODE2 and $f_i(z)$ and $f_j(z)$ provide for a Gauss hypergeometric recurrence relation we can expect ladder operations and quantization. In one of his celebrated papers [11] Schrödinger took exactly one of the hypergeometric pull-back function we found applying the Schwarz criteria in equation (16) and showed how to factorize with this transcendental transformation the Gauss hypergeometric ODE2. He was aware that most of the interesting functions occurring in physics are either special or limiting cases of Gauss's function and indicated with $z \mapsto \phi(\theta) = \cos^2(\theta/2) = \cos^2(\omega_q\alpha)$ a quadruple of factorizations of the hypergeometric equation. In his transformation $t = \cos\theta = 2\phi - 1$ it was important that $0 \le \phi \le 1$ and that the independent variable θ is restricted to $\pi \ge \theta \ge 0$. He then found that

$$\frac{d^2f}{d\theta^2} + \left(\frac{\omega_1\cos\theta + \omega_2}{\sin\theta}\right)\frac{df}{d\theta} + \omega_q^2 f = 0,$$

is factorizable and related to the hypergeometric equation in eq.(9) by

$$\omega_1 = a+b, \qquad \omega_2 = a+b+1-2c, \qquad \omega_q^2 = -ab,$$

which provides for a direct route to the Schrödinger equation and associated potential with quantization, factorization and corresponding ladder operators [1], [14], [11]. With $\omega_1 = \pm\omega_2$, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, and $\theta = 2\omega_q\alpha$ our spherical solutions with $q = \omega_q^2 = \epsilon^2 - \lambda^2 < 0$ correspond to the Gegenbauer polynomials $C_n^\lambda(t)$ with index $n = \pm\epsilon - \lambda$

$$\xi(t) = C_n^\lambda(t) = C_{\pm\epsilon-\lambda}^\lambda(t), \tag{20}$$

(see 8. 938 of Ref. [263]) with

$$(1-t^2)\frac{d^2\xi(t)}{dt^2} - t(1+2\lambda)\frac{d\xi(t)}{dt} + \left(\epsilon^2 - \lambda^2\right)\xi(t) = 0\ , \tag{21}$$

related to the Gauss hypergeometric function

$$C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)\Gamma(2\lambda)}\ {}_2\mathrm{F}_1\left(\begin{matrix}2\lambda+n,\ -n\\ \frac{1}{2}+\lambda\end{matrix}\,\middle|\,\frac{1-t}{2}\right), \tag{22}$$

or

$$C_{\pm\epsilon-\lambda}^\lambda(t) = \frac{\Gamma(\lambda\pm\epsilon)}{\Gamma(\pm\epsilon-\lambda+1)\Gamma(2\lambda)}\ {}_2\mathrm{F}_1\left(\begin{matrix}\lambda\pm\epsilon,\ \lambda\mp\epsilon\\ \frac{1}{2}+\lambda\end{matrix}\,\middle|\,\frac{1-t}{2}\right). \tag{23}$$

The associated Legendre polynomials $P_\nu^\mu(t)$ (spherical harmonics) are with $\mu = \frac{1}{2} - \lambda$, $\nu = \pm\epsilon + \frac{1}{2}$, $n = \nu + \mu - 1$ also related to Gauss hypergeometric function

$$P_\nu^\mu(t) = P_{-\nu-1}^\mu(t) = \frac{1}{\Gamma(1-\mu)}\left(\frac{t+1}{t-1}\right)^{\frac{\mu}{2}}\ {}_2\mathrm{F}_1\left(\begin{matrix}-\nu,\ \nu+1\\ 1-\mu\end{matrix}\,\middle|\,\frac{1-t}{2}\right), \tag{24}$$

where the direct relation between $C_n^\lambda(t)$ and $P_\nu^\mu(t)$ is given by

$$C_n^\lambda(t) = \frac{\Gamma(2\lambda+n)\Gamma(\lambda+\frac{1}{2})}{\Gamma(n+1)\Gamma(2\lambda)}\left(\frac{2}{1-t^2}\right)^{(\lambda-\frac{1}{2})/2} P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(t)\ . \tag{25}$$

For positive ϵ we can assign to the quantum numbers a holonomy contribution according to eq.(11), which are metric distortions given by two hypergeometric fields superimposing nonlinearly to one hypergeometric field representing the total holonomy. The $P_\nu^\mu(t)$ obey with pull-back $z = \sin^2(\theta_s/2) = (1-t)/2$ and $r = \tan(\theta_s/2) = \tanh(\theta_h/2)$ the wave equation

$$\frac{d}{dr}\left[\left(1-r^2\right)\frac{dP_\nu^\mu(r)}{dr}\right] + \left[\nu(\nu+1) - \frac{\nu^2}{1-r^2}\right]P_\nu^\mu(r) = 0, \tag{26}$$

leading in the hyperbolic case to the Schrödinger equation of MAP on the disk

$$\frac{d^2\Psi_\nu^\mu(\theta_h)}{d\theta_h^2} + \left[\frac{D}{\cosh^2\theta_h} - \mu^2\right]\Psi_\nu^\mu(\theta_h) = 0, \tag{27}$$

where μ^2 is the energy eigenvalue. The Pöschl-Teller (PT) type potential

$$V(\theta_h) = -\frac{D}{\cosh^2\theta_h} \tag{28}$$

has potential depth $D = \nu(\nu+1)$. From a dynamical group approach there is an explicit connection of the PT potential with the $su(1,1)$ and $su(2)$ algebra describing the bending of coordinates in molecules, where θ_h gives the relative deviation from the equilibrium position [7].

6 Generalized Gauss Linking Number Concept

Thomas precession is the spatial rotation from relativistic "boosts" around a curve and corresponds to the hyperbolic projection onto the disk. The corresponding spin current induced by relativistic motion is for particle physics a spin factor. Such a spin or phase factor can appear as a Wilson loop in Chern-Simons gauge theory. The quantification of the loop current is how to count the Writhe that is equal to the Linking minus the Twist. The Writhe can be assigned to the Gauss linking integral, the twist to the integral number of extra loops. Historically, the Gauss linking concept was limited to flat space but has been recently extended to higher-dimensional spherically curved surfaces, which provides for a generalized (curved) electromagnetism, Biot-Savart law, and Maxwell's equations (DeTurck and Gluck, Kupferberg [8]). It is easy to see, that the Gauss linking differential form generalized to higher-dimensional curved surfaces fits perfectly to our Fuchsian differential form and provides with the proper pull-back for the same solutions. According to Kupferberg [8], if the SO(a,b)-invariant differential form

$$\omega = \phi(t)\boldsymbol{x}\wedge\boldsymbol{y}\wedge d\boldsymbol{x}^{\wedge a-1}\wedge d\boldsymbol{y}^{\wedge b-1}, \quad t = \boldsymbol{x}\cdot\boldsymbol{y}, \tag{29}$$

on $H^+\times H^-$ in $\mathbf{R}^{(a,b)}$ has $d_y d_x\omega = 0$, it satisfies the ordinary differential equation (13) with p-functions according to Table 1, e.g., for $f(\alpha) = \phi[t(\alpha)]$ with pull-back $t = \sinh\alpha$ and $p = (a+b)\tanh\alpha$ the corresponding ODE's are given by

$$f'' + (a+b)(\tanh\alpha)f' + abf = 0,$$

$$(\cosh\alpha)^2\phi'' + (a+b+1)(\sinh\alpha)\phi' + ab\phi = 0.$$

7 Quantum Chaotic Transitions

We are not only interested in the MAP condition but also in the dynamics of bifurcations or limit cycles given by the cosine map, see Figure 1. The hypergeometric ODE2 solutions to model the chaotic spin exchange via precession could be extended to distortion by local coupling singularities. In

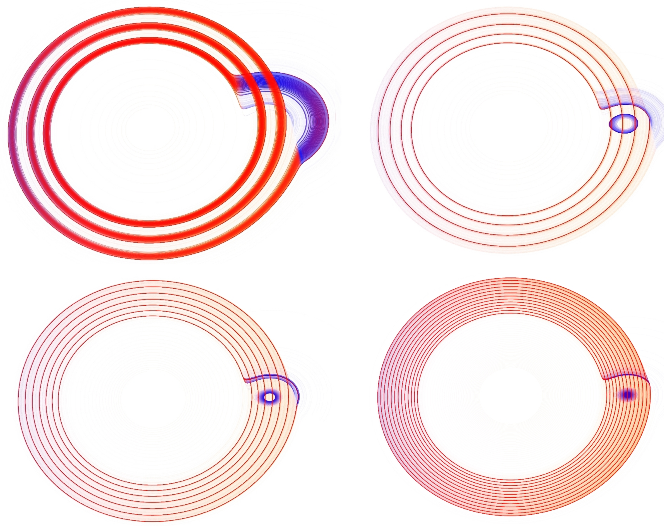

Fig. 3. A Chua-type model for chaotic quantum transitions, where a local singularity triggers the dissipation of a spin quantum characterized by a winding number load for $j = M$. On the top left we have $k/M = 2$, on the top right we have $k/M = 3$, on the bottom left we have $k/M = 5$, on the bottom right we have $k/M = 8$. The blue color indicates the influence of the holonomic coupling term $\cos[j(y - z)]$.

physics MAP can represents a special chaotic behavior in the precession angle if the rotor angular velocity is linearly coupled by (an)holonomy to the precession angular velocity and angle [4]. With metric distortion proportional to precession (precession angular velocity in a transport process), and precession proportional to spin, energy proportional to spin would recurrently conform Einstein's proposal that energy is proportional to a metric distortion. The linear coupling provides for conic paths and allows spinning up and controlling the rotor simply by forcing precession at special quantum magic precession angles. To approximate this behavior with monopole coupling strength $M\theta$ and nonlinear holonomic control function $\pi \cos[j(y - z)]$ from gyroscopic precession, we refer to a well known and rather simple system of coupled differential equations [3], where the geodesic flow of the attractor can be simulated and conceptually approached by a Chua-type system [12], a well known and real-world model of chaotic dynamics given by

$$\frac{dz}{dt} = M(y - z) - \pi \cos[j(y - z)] \ ,$$

$$\frac{dy}{dt} = x - M(y - z) \ ,$$

$$\frac{dx}{dt} = -y + \frac{x}{k} \ . \tag{30}$$

Depending on the source strength $1/k$ of the driving oscillator and the number of phase singularities given by j/M, MAP shows a characteristic quantum spin dissipation dynamics given by a kind of winding ratio k/M, which is coupling strength $\propto 1/M$ divided by source strength $1/k$. For $j = M$ the phase space dynamics is shown in Figure 3. The differential equation system eqs.(30) describe chaotic currents, where the nonlinear impedance in Chua's original electronic circuit controlling the linear oscillator is a transcendental function $f(z) = \pi \cos[j(y - z)]$, the holonomic coupling term given by the total phase minus the Berry geometric phase [4],[3]. MAP can be found in the z-singularity $dz/dt = 0$ with $\theta = y - z$ and $r_s = \pi j$ and $\theta = j\alpha$, which can be approached and illustrated if we take the precession angle θ as the gauge potential term (showing Coulomb type "charge" and dipole effects) and the rotor spin as the electric current. Both systems have 3 degrees of freedom (two voltage y, z, one current x) and 3 energy storage elements (two capacitors and one inductivity as the rotor angular momentum setting the timescale), see [3]. In both cases a linear oscillator (precession in MAP) is coupled to a nonlinear element (holonomy in MAP). The nonlinear element responsible for chaos and bifurcation is driven by precession, where the geometric coupling current is delivered by the conductivity term providing for j missing or extra loops. The geometric phase induced by the curved path of the rotor or external curvature and part of the coupling increases with precession angle. Limitations and instabilities of the spin current exchange can be assigned to geometric phase bifurcations at high precession loads as the driving gauge potential.

8 Conclusions

We have considered triangle maps of spin/precession vector relations, where the dynamics in spaces of different curvature can be connected by a chaotic MAP state, whichs show a chaotic precession dynamics at high precession angle values. The MAP singularities can be assigned to a (composite) hypergeometric mapping function composed by dual hypergeometric functions related by inversion. Thus, MAP becomes a hypergeometric quantum solution describing quantized metric distortions providing for holonomy eigenstates characterized by winding numbers. Interesting applications for this new approach can be found in atoms or solids forcing a spin dynamics on closed loops or a spatially periodic holonomy. This finding could be relevant to the understanding of frictionless or lossless (no radiation) nuclear, atomic, solid state quantum spin transitions, vertex flows, and large scale gravitational anomalies observed with rotated rotators like satellites.

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